8.1.8

If T is the triangle with vertices[1] $(0,0)$, $(3,0)$ and $(3,4)$, we are asked to compute

$$\int_{\partial T} F \cdot ds = \int\int_T Q_x - P_y$$

Which the equality comes from Theorem 3. This gives

$$\int_0^3 \int_0^{4x/3} 10y - 8y\ dy\ dx = \int_0^3 \left(\frac{4x}{3}\right)^2\ dx = 16$$

8.1.18

By picking $P = \frac{\partial f}{\partial y}$ and $Q = -\frac{\partial f}{\partial x}$ we can apply Green's Theorem and

$$\int_{\partial D} P\ dx + Q\ dy = \int\int_D Q_x - Py\ dA =$$

$$= \int\int_D -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\ dA = \int\int_D 0\ dA = 0$$

8.1.23

Parametrizing $x = a\cos t, y = b\sin t$, and hence $dx = -a\sin t\ dt$ and $dy = b\cos t\ dt$ Theorem 2 yields

$$A = \frac{1}{2}\int_{\partial D} x\ dy - y\ dx = \int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\ dt =$$

$$= \frac{1}{2}\int_0^{2\pi} ab\ dt = ab\pi$$

8.2.16

Green's Theorem states that the integral of $\nabla \times F$ over a surface S is the integral of F over its boundary. Hence if two surfaces S and S' have the

[1] Actually the problem does not specify the origin, so we shall suppose it is $(0,0)$

same boundary, the previous integral is the same. In our case it is easier to integrate it over the portion of the plane inside the circle, since $\nabla \times F \equiv (-2,-2,-2)$, which is $-\sqrt{12}$ times the normal unit vector pointing outwards. This gives that the integral is just $-\sqrt{12}$ times the area of the circle.

The distance from the plane to the origin is given by the distance point-plane formula, which yields $\frac{1}{\sqrt{3}}$ in our case. Hence the radius of the circle by the Pythagorean Theorem is $\frac{\sqrt{2}}{\sqrt{3}}$ and thus the area is

$$A = \frac{2}{3}\pi$$

And the integral gives $\frac{-2\sqrt{12}\pi}{3}$.

8.2.25

If we draw a closed loop C over the surface, the loop separates it into two surfaces S_1, S_2, which have as boundary C but with different orientations.

Then by Green Theorem

$$\int\int_S \nabla \times F \cdot dS = \int\int_{S_1} \nabla \times F \cdot dS + \int\int_{S_2} \nabla \times F \cdot dS =$$

$$= \int_{C^+} F \cdot ds + \int_{C^-} F \cdot ds = 0$$

8.3.14

If we parametrize $x = \cos t$, $y = \sin t$, $z = 0$ we have

$$\int_C F \cdot ds = \int_0^{2\pi} (\cos t \sin t, 0, 0) \cdot (-\sin t, \cos t, 0)\ dt = \int_0^{2\pi} -\sin^2 t \cos t\ dt$$

Which is a symmetric function around 0. Hence the integral vanishes and the circulation is 0.

8.3.16

(a) Since the value of $x^2 + y^2$ is 1, we have that the integral is the same as

$$\int_C x\ dy - y\ dx$$

Which is two times the area of C, which is 2π.

(b)The integral of F over the unit circle is nonzero, which contradicts part (i) of Theorem 7.

(c) We have

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

But the corollary doesn't have to hold, since F is not C^1 in all $\mathbb{R}^2$. In fact it is not defined at $(0, 0)$.

8.4.6

Let B be the unit ball, which has ∂B as the unit sphere. Then by Gauss's Theorem we have

$$\int\int_{\partial B} F \cdot dS = \int\int\int_B \nabla \cdot F\ dV = \int\int\int_B 3(x^2 + y^2 + z^2)\ dx\ dy\ dz$$

In spherical coordinates we have $(x^2 + y^2 + z^2) = \rho^2$ and it is well known that the Jacobian of the change is $\rho^2 \sin\phi$. Then we have that the integral is just

$$\int_0^\pi \int_0^{2\pi} \int_0^1 3\rho^4 \sin\phi\ dr\ d\theta\ d\phi = \frac{12}{5}\pi$$

8.4.7

Let us compute the integral over the top face S, which can be parametrized by $(u, v, 1)$ with $0 \leq u, v \leq 1$. Then we have $T_u\ timesT_v = (0, 0, 1)$ and hence

$$\int\int_S F \cdot dS = \int_0^1 \int_0^1 (u, v, 1) \cdot (0, 0, 1)\ du\ dv = 1$$

Now if we parametrize the bottom face as $(v, u, 1)$ (the normal vector has to point outwards!) we get $T_u \times T_v = (0, 0, -1)$ and in this case

$$\int\int_S F \cdot dS = \int_0^1 \int_0^1 (u, v, 0) \cdot (0, 0, 1)\ du\ dv = 0$$

Analogously, we get 1 and 0 for each other pair of opposite faces, and thus the total integral is $1+0+1+0+1+0=3$.

We can check this result directly by using Gauss Theorem: If C is the unit cube then

$$\int\int_{\partial C} F \cdot dS = \int\int\int_C \nabla \cdot F \; dV = \int\int\int_C 3 \; dV$$

Which is three times the volume of the cube. This gives 3 as desired.

8.4.16

Since $F \cdot n \; dA = F \cdot dS$ we can use Gauss's Theorem and hence

$$\int\int_{\partial S} F \cdot n \; dA = \int\int\int_S \nabla \cdot F \; dV = \int\int\int_S (x^2+y^2)^2 \; dV$$

By plugging cylindrical coordinates we have $x^2+y^2=r^2$ and that the determinant of the Jacobian of the change is r. This gives that the integral can be transformed to

$$\int_0^1 \int_0^{2\pi} \int_0^1 r^5 \; dr \; d\theta \; dz = \frac{\pi}{3}$$